

M E T U
Northern Cyprus Campus

Math 219 Differential Equations I. Exam 27.03.2016					
Last Name:	Dept./Sec. : Name : Time : 09: 40 Student No: Duration : 100 minutes			Signature	
6 QUESTIONS ON 4 PAGES					TOTAL 110 POINTS
1	2	3	4	5	6

Q1 (20 pts.) Find the general solution to the differential equation $\frac{dy}{dt} = -\frac{4t+3y}{2t+y}$.

This is a homogeneous differential equation.

$$\frac{dy}{dt} = -\frac{4+\frac{3y}{t}}{2+\frac{y}{t}}$$

$$\text{Let } v = \frac{y}{t}$$

$$v + tv' = -\frac{4+3v}{2+v}$$

$$tv' = -\frac{4+3v-v}{2+v}$$

$$tv' = \underline{-4-3v-2v-v^2}$$

$$tv' = -\frac{4+5v+v^2}{2+v}$$

$$\frac{2+v}{(v+1)(v+4)} = \frac{A}{v+1} + \frac{B}{v+4}$$

$$2+v = A(v+4) + B(v+1)$$

$$v=-1 \quad 1 = A \cdot 3 \Rightarrow A = \frac{1}{3}$$

$$v=-4 \quad -2 = B(-3) \Rightarrow B = \frac{2}{3}$$

$$\int \frac{2+v}{v^2+5v+4} dv = \int \frac{1/3}{v+1} dv + \int \frac{2/3}{v+4} dv$$

$$= \frac{1}{3} \ln|v+1| + \frac{2}{3} \ln|v+4|$$

\Rightarrow Eqn's ^{general} solution is:

$$\frac{1}{3} \ln|v+1| + \frac{2}{3} \ln|v+4| = -\ln|t| + C$$

$$\Rightarrow \frac{1}{3} \ln|y+x| + \frac{2}{3} \ln|y+4x| = C$$

Use partial fractions!

$$\int \frac{2+v}{v^2+5v+4} dv = \int \frac{2+v}{(v+1)(v+4)} dv$$

$$(y+x)^{1/3} | y+4x|^{2/3} = A \neq 0$$

By observation $y=-x$ &
 $y=-4x$
 are also solutions!

Q2 (20 pts.) Use the integrating factors to solve $y' + (1/t)y = 2 \sin(2t)$, $t > 0$.

Integrating factor for this linear eqn is

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$

Solution of DE

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \int \mu(t) 2 \sin 2t dt \\ &= \frac{1}{t} \int t 2 \sin 2t dt \quad (\text{Apply integration by parts}) \\ &= \frac{1}{t} \left\{ -t \cos 2t - \int -\cos 2t dt \right\} \quad t = u \rightarrow du = dt \\ &\quad dv = 2 \sin 2t dt \rightarrow v = -\cos 2t \\ &= \frac{1}{t} \left\{ -t \cos 2t + \frac{\sin 2t}{2} + C \right\} \\ y(t) &= -\cos 2t + \frac{\sin 2t}{2t} + C \cdot \frac{1}{t} \end{aligned}$$

Q3 (20 pts.) A tank contains 120 L of water and 260 gr of salt. Water containing a salt concentration of $2 + e^t \cos(5t)$ lb/L flows into the tank at a rate of 5 l/min, and the mixture in the tank flows out at the same rate. Write down the IVP to find the amount $Q(t)$ of salt in the tank at any time t . (Do not solve the IVP).

Volume of water is 120 lt.

$$\frac{dQ}{dt} = (2 + e^t \cos 5t) \cdot 5 - 5 \cdot \frac{Q}{120}$$

$$Q(0) = 260 \text{ gr}$$

Q4 (20 pts.) Find an integrating factor μ either in t or y , and solve the following nonlinear differential equation $1 + \left(\frac{t}{y} - \cos(y)\right)y' = 0$.

This equation is not exact:

$$dt + \left(\frac{t}{y} - \cos y\right) dy = 0$$

TRY: $\mu = \mu(y)$.

$$\underbrace{\mu(y) dt}_{P} + \underbrace{\mu(y) \left(\frac{t}{y} - \cos y\right) dy}_{Q} = 0$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial \mu}{\partial y} = \frac{\partial Q}{\partial t} = \frac{\mu(y)}{y}$$

$$\Rightarrow \frac{d\mu}{dy} = \frac{\mu}{y}$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{dy}{y}$$

$$\Rightarrow \boxed{\mu = y}$$

Multiply by $\mu(y) = y$

$$\underbrace{y dt}_M + \underbrace{(t - y \cos y) dy}_N = 0$$

$$\frac{\partial F}{\partial t} = \underbrace{y}_M \Rightarrow F = t \cdot y + g(y)$$

$$\frac{\partial F}{\partial y} = t + g'(y) = \underbrace{t - \cos y}_N$$

$$\Rightarrow g'(y) = -\cos y$$

$$\Rightarrow g(y) = -\sin y$$

Potential fun: $F = ty - \sin y$

Solution:

$$\boxed{ty - \sin y = C}$$

Q5 Bonus (10 pts.) Consider the IVP $y' = \frac{8+t^3}{3y-y^2}$, $y(2) = 1$. State where in ty -plane (a rectangular region) the hypotheses of Existence and Uniqueness Theorem are satisfied.

$f(t, y) = \frac{8+t^3}{3y-y^2} \Rightarrow f(t, y)$ is defined and continuous on $\mathbb{R}^2 \setminus \{y=3 \text{ or } y=0\}$

$$\frac{\partial f}{\partial y} = (8+t^3)(-1)(3y-y^2)^{-2}(3-2y)$$

$$= -\frac{(8+t^3)(3-2y)}{(3y-y^2)^2} \Rightarrow \frac{\partial f}{\partial y} \text{ is defined and continuous on}$$

$\uparrow y$

$y=3$

$y=0$

$$R = (-\infty, \infty) \times (0, 3)$$

$$\mathbb{R}^2 \setminus \{y=3 \text{ or } y=0\}$$

satisfies all requirements of the existence and uniqueness theorem.

Q6 (20 pts.) Solve the following nonhomogeneous linear algebraic system

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ -x_1 + 2x_2 - 4x_3 + 2x_4 = 0 \\ x_1 + x_3 + 3x_4 = -1 \end{cases}$$

using the reduced echelon form. Write down the solutions in the form $Sol_{nh} = Sol_h + \mathbf{y}$, where \mathbf{y} is a special solution to nonhomogeneous linear system.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ -1 & 2 & -4 & 2 & 0 \\ 1 & 0 & 1 & 3 & -1 \end{array} \right] \xrightarrow{\substack{R_2 + R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3}} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 1 & -1 & 3 & -2 \end{array} \right]$$

$$R_3 - R_2 \rightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{array} \right] \rightarrow \begin{aligned} x_1 &= 1 + x_2 - 2x_3 = 1 + (-5 - 4x_4) - 2(-3 - x_4) \\ x_2 &= +1 + 2x_3 - 2x_4 = +1 - 6 - 2x_4 - 2x_4 \\ x_3 &+ x_4 = -3 \Rightarrow x_3 = -3 - x_4 \\ x_4 &= \text{free} = x_4 \end{aligned}$$

$$x_1 = 2 + -2x_4$$

$$x_2 = -5 + -4x_4$$

$$x_3 = -3 + -x_4$$

$$x_4 = 0 + x_4$$

$$x = \underbrace{\begin{bmatrix} -2 \\ -4 \\ -1 \\ 1 \end{bmatrix}}_{\sim} \circ x_4 + \begin{bmatrix} 2 \\ -5 \\ -3 \\ 0 \end{bmatrix}$$

$$Sol_{nh} = Sol_h + \mathbf{y}$$